

Determination of α_s from τ decays

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Hadronic τ decays offer the possibility of determining the strong coupling α_s at relatively low energy. Precisely for this reason, however, good control over the perturbative QCD corrections, the non-perturbative condensate contributions in the framework of the operator product expansion (OPE), as well as the corrections going beyond the OPE, the duality violations (DVs), is required. On the perturbative QCD side, the contour-improved versus fixed-order resummation of the series is still an issue, and will be discussed. Regarding the analysis, self-consistent fits to the data including all theory parameters have to be performed, and this is also explained in some detail. The fit quantities are moment integrals of the τ spectral function data in a certain energy window and care should be taken to have acceptable perturbative behaviour of those moments as well as control over higher-dimensional operator corrections in the OPE.

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1. Introduction

The τ lepton is the only known lepton heavy enough to also decay hadronically. In view of its mass of $M_\tau \approx 1.8 \text{ GeV}$, it provides an excellent laboratory for the investigation of low-energy QCD. On the other hand, the scale is so low that the inclusion of effects beyond perturbative QCD is required for precision studies. The non-perturbative effects appear in the framework of the operator product expansion (OPE) as vacuum-condensate terms, or contributions beyond the OPE, named *duality violations*, since they signal the breakdown of the quark-hadron duality picture of the OPE.

The central experimental observable is the total τ hadronic width

$$R_\tau \equiv \frac{\Gamma[\tau^- \rightarrow \text{hadrons } \nu_\tau(\gamma)]}{\Gamma[\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma)]} = 3.6280(94) [1], \quad (1.1)$$

which was explored in the seminal theoretical analyses [2, 3, 4] as a means to determine the strong coupling α_s . In subsequent years, the acquisition of large data sets by the ALEPH and OPAL collaborations allowed the extraction of inclusive spectral τ -decay distributions, the so-called spectral functions, and their separation into the light-quark (up and down) vector and axialvector channels, as well as the strange channel [5, 6, 7]. The additional experimental information can be employed to compute moments of the decay spectra which yield further data points to be compared to theory. In principle, also exclusive decay distributions are available and could be investigated, though they are less useful for basic QCD studies.

On the theoretical side, recent years have seen several improvements in the description of hadronic τ decays. The most important one was the impressive analytical computation of the perturbative order α_s^4 correction [8], which has revived the interest in α_s analyses from τ decays. Generally, one might think that an additional order in the perturbative expansion would reduce the theoretical uncertainties. However, the theoretical expectation for the total hadronic width and moments of the decay spectra depends on the way that large logarithms appearing in the expansion are resummed by means of the renormalisation group (RG). The two most commonly used approaches are fixed-order perturbation theory (FOPT), and contour-improved perturbation theory (CIPT) [9, 10], and the newly available $\mathcal{O}(\alpha_s^4)$ correction has made the dispersion in the theoretical prediction only more distinct. Hence, efforts in recent years have been devoted to better understand the origin of the differences in the RG improvement and attempts to single out the more reliable procedure [11, 12, 13, 14, 15, 16]. Those shall be reviewed in more detail below.

The second main step forward has been a proper inclusion of violations of quark-hadron duality in the τ sum rule analysis. The calculation of the theoretical side of the sum rule involves an integral over a circle in the complex energy plane with radius $s_0 = M_\tau^2$ or smaller (down to about 1.5 GeV^2). In the region close to the physical axis with real, positive s_0 , the OPE breaks down and contributions beyond it become relevant. This should be clear, as the OPE is unable to directly describe the hadronic resonance structure. Since a sound theoretical description of duality violations (DVs) is not available, a initial model was laid out in ref. [17], and later refined in refs. [18, 19]. A complete analysis requires the simultaneous, self-consistent determination of all occurring parameters, that is, α_s , the OPE condensate parameters, as well as the DV model parameters. On the basis of the OPAL data, such an analysis was performed in refs. [20, 21], the discussion of which will be the second main topic of this writeup.

2. The anatomy of $R_{\tau,V+A}$

Most suitable for a determination of α_s are the τ decay rates into light u and d quarks $R_{\tau,V/A}$ via the vector or axialvector current, and the related moments, since in this case power corrections are especially suppressed. Theoretically, $R_{\tau,V/A}$ can be expressed as

$$R_{\tau,V/A} = \frac{N_c}{2} |V_{ud}|^2 S_{\text{EW}} \left[1 + \delta^{(0)} + \sum_{D \geq 2} \delta_{ud,V/A}^{(D)} + \delta_{V/A}^{\text{DV}} \right], \quad (2.1)$$

where $S_{\text{EW}} = 1.0201(3)$ [22, 23, 24] comprises electroweak corrections, $\delta^{(0)}$ denotes the perturbative QCD correction, the $\delta_{ud,V/A}^{(D)}$ are quark mass and higher D -dimensional operator corrections which arise in the framework of the OPE, and $\delta_{V/A}^{\text{DV}}$ is the DV contribution beyond the OPE. Before entering a more detailed discussion of the α_s analysis, let us review our current knowledge regarding the anatomy of $R_{\tau,V+A}$, which takes the general form

$$R_{\tau,V+A} = 3 |V_{ud}|^2 S_{\text{EW}} \left[1 + \delta^{(0)} + \delta_{V+A}^{\text{NP}} \right]. \quad (2.2)$$

The higher-dimensional OPE and the DV corrections have been lumped together into the non-perturbative correction δ_{V+A}^{NP} . To obtain the general picture, first we assume α_s to be known. Evolving the present PDG average $\alpha_s(M_Z) = 0.1184(7)$ to the τ mass, yields $\alpha_s(M_\tau) = 0.3186(58)$. Even though the PDG average also includes α_s from τ 's, removing this datum only has a small influence, so that it appears justified to employ the PDG value.

To derive our expectation for $\delta^{(0)}$, which gives the dominant correction to $R_{\tau,V+A}$, the theoretical expressions in FOPT and CIPT are briefly reminded. In FOPT the fixed renormalisation scale $\mu = M_\tau$ is chosen, which results in [11]

$$\delta_{\text{FO}}^{(0)} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^n k c_{n,k} J_{k-1}, \quad (2.3)$$

where $a(\mu^2) \equiv \alpha_s(\mu)/\pi$, and $c_{n,k}$ are coefficients which appear in the perturbative expansion of the vector correlation function. At each perturbative order, the coefficients $c_{n,1}$ can be considered independent, while all other $c_{n,k}$ with $k \geq 2$ are calculable from the RG equation. Explicit results for the analytically known $c_{n,1}$ can be found in ref. [8]. Furthermore, the J_l are contour integrals in the complex s -plane which for example are presented in [11].

In CIPT, on the other hand, an s -dependent renormalisation scale is introduced, which partially resums higher-order effects, namely the ones due to the running of α_s , leading to [9, 10]

$$\delta_{\text{CI}}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(M_\tau^2) \quad (2.4)$$

in terms of the contour integrals $J_n^a(M_\tau^2)$ over the running coupling, defined as:

$$J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-M_\tau^2 x). \quad (2.5)$$

In contrast to FOPT, for CIPT each order n only depends on the corresponding coefficient $c_{n,1}$. All contributions proportional to the coefficient $c_{n,1}$, which in FOPT appear at all perturbative orders equal or greater n , are resummed into a single term.

Numerically, the two approaches lead to significant differences. Employing the value for $\alpha_s(M_\tau)$ given above, one obtains

$$\delta_{\text{FO}}^{(0)} = 0.2022 \pm 0.0069 \pm 0.0030 = 0.2022(75), \quad (2.6)$$

$$\delta_{\text{CI}}^{(0)} = 0.1847 \pm 0.0048 \pm 0.0033 = 0.1847(58), \quad (2.7)$$

where the first error corresponds to the uncertainty in $\alpha_s(M_\tau)$ and the second to an estimate of higher orders through a variation of the coefficient $c_{5,1} = 283 \pm 283$ [12]. Given the results for $\delta^{(0)}$, the value $|V_{ud}| = 0.97425(22)$ [26], as well as $R_{\tau,V+A} = 3.4671(82)$, which follows from $B_{VA} = 61.85(11)\%$ and $B_e^{\text{uni}} = 17.839(28)\%$ [1], one can estimate δ_{V+A}^{NP} , with the finding

$$\delta_{V+A,\text{FO}}^{\text{NP}} = -0.0086(80), \quad \delta_{V+A,\text{CI}}^{\text{NP}} = 0.0089(65). \quad (2.8)$$

This analysis shows that δ_{V+A}^{NP} is expected to be $\lesssim 1\%$ and from this simple estimate compatible with zero at about 1σ . Hence, the non-perturbative correction is much smaller than the perturbative one. Still, at the current level of precession, where the error on the PDG average for α_s induces a shift in $\delta^{(0)}$ of roughly 0.5% , the non-perturbative contribution becomes relevant. Before real progress in the determination of α_s from τ decays can be made, however, the difference between FOPT and CIPT has to be settled first, as it is certainly more important.

3. Adler function at higher orders

The question whether FOPT or CIPT provides a better approximation to $\delta^{(0)}$ hinges on the behaviour of the vector correlator, or equivalently the Adler function, at higher orders. To make progress in this direction, additional information beyond the analytically known orders has to be taken into account. An attempt towards this goal, based on a model for the Borel-transformed Adler function, was presented in ref. [12] and shall be described next.

The perturbative expansion of the Adler function $D(s)$ takes the form

$$4\pi^2 D(s) \equiv 1 + \widehat{D}(s) = \sum_{n=0}^{\infty} c_{n,1} a(s)^n. \quad (3.1)$$

For the following it is slightly more convenient to utilise the function $\widehat{D}(s)$ instead of $D(s)$. Its Borel transform $B[\widehat{D}](t)$ is defined by

$$\widehat{D}(\alpha) \equiv \int_0^{\infty} dt e^{-t/\alpha} B[\widehat{D}](t). \quad (3.2)$$

The integral $\widehat{D}(\alpha)$, if it exists, gives the Borel sum of the original divergent series. It was found that the Borel-transformed Adler function $B[\widehat{D}](t)$ obtains infrared (IR) and ultraviolet (UV) renormalon poles at positive and negative integer values of the variable $u \equiv 9t/(4\pi)$, respectively [27, 28, 29]. (With the exception of $u = 1$, since there is no corresponding gauge-invariant $D = 2$ operator.)

Guided by the large- β_0 approximation [29], the influence of renormalon poles on the perturbative expansion should be as follows: asymptotically, that is for high orders, the perturbative expansion is dominated by the $u = -1$ UV pole that is closest to $u = 0$. At intermediate orders

some dominance of the low-lying IR poles ($u = 2, u = 3$) is observed, while very low orders are no longer dominated by a small subset of renormalon poles. This suggests the following ansatz [12]

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\text{UV}}](u) + B[\widehat{D}_2^{\text{IR}}](u) + B[\widehat{D}_3^{\text{IR}}](u) + d_0^{\text{PO}} + d_1^{\text{PO}}u, \quad (3.3)$$

which includes one UV renormalon at $u = -1$, the two leading IR renormalons at $u = 2$ and $u = 3$, as well as polynomial terms for the two lowest perturbative orders. Explicit expressions for the UV and IR renormalon pole terms $B[\widehat{D}_p^{\text{UV}}](u)$ and $B[\widehat{D}_p^{\text{IR}}](u)$ can be found in section 5 of ref. [12].

Apart from the residues d_p^{UV} and d_p^{IR} , the full structure of the renormalon pole terms is dictated by the OPE and the RG. Therefore, the model (3.3) depends on five parameters, the three residues d_1^{UV} , d_2^{IR} and d_3^{IR} , as well as the two polynomial parameters d_0^{PO} and d_1^{PO} . These parameters can be fixed by matching to the perturbative expansion of $\widehat{D}(s)$ up to $\mathcal{O}(\alpha_s^5)$, whereby also the estimate for $c_{5,1} = 283$ is required. The parameters of the model (3.3) then are found to be:

$$\begin{aligned} d_1^{\text{UV}} &= -1.56 \cdot 10^{-2}, & d_2^{\text{IR}} &= 3.16, & d_3^{\text{IR}} &= -13.5, \\ d_0^{\text{PO}} &= 0.781, & d_1^{\text{PO}} &= 7.66 \cdot 10^{-3}. \end{aligned} \quad (3.4)$$

The fact that the parameter d_1^{PO} turns out small implies that the coefficient $c_{2,1}$ is already reasonably well described by the renormalon pole contribution, although it was not used to fix the residues.

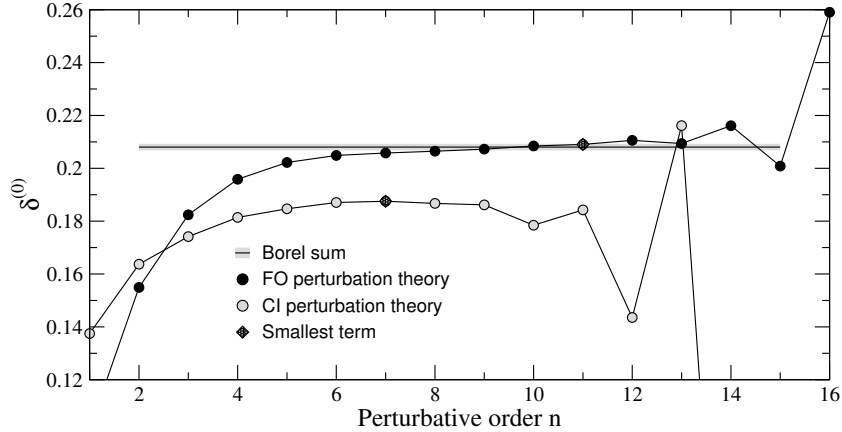


Figure 1: Results for $\delta_{\text{FO}}^{(0)}$ (full circles) and $\delta_{\text{CI}}^{(0)}$ (grey circles) at $\alpha_s(M_\tau) = 0.3186$, employing the model (3.3), as a function of the order n up to which the terms in the perturbative series have been summed. The straight line represents the result for the Borel sum of the series.

The implications of the model (3.3) for $\delta^{(0)}$ in FOPT and CIPT is graphically represented in figure 1. The full circles denote the result for $\delta_{\text{FO}}^{(0)}$ and the grey circles the one for $\delta_{\text{CI}}^{(0)}$, as a function of the order n up to which the perturbative series has been summed. The straight line corresponds to the principal value Borel sum of the series. The order at which the series have their smallest terms is indicated by the grey diamonds. As is evident from figure 1, FOPT displays the behaviour expected from an asymptotic series: the terms decrease up to a certain order around which the closest approach to the resummed result is found, and for even higher orders, the divergent large-order behaviour of the series sets in. For CIPT, on the other hand, the asymptotic behaviour sets in

earlier, and the series is never able to come close to the Borel sum. In the large- β_0 approximation, this was already observed in ref. [30].

The superiority of FOPT over CIPT critically depends on the size of the $u = 2$ residue d_2^{IR} . If this residue in real QCD would turn out substantially smaller than in eq. (3.4), CIPT could provide the better approach to the resummed series. Corresponding models have been studied in ref. [16] whose main aim was to investigate the perturbative behaviour of a large set of moments used in α_s analyses from τ decays within the Borel models. Nevertheless, there is no known mechanism in QCD to suppress the $u = 2$ residue and thus the behaviour that favours FOPT appears more likely. Other studies that implement information on the renormalon structure of QCD include refs. [13, 15], where also conformal mappings were applied to the series.

4. Self-consistent fits, OPE and duality violations

A self-consistent analysis of α_s from τ decays includes the simultaneous determination of all other parameters entering the game. These include QCD condensates as well as DV parameters. The required additional information can be obtained through the use of weighted integrals of the inclusive decay spectra up to an energy $s_0 \leq M_\tau^2$, the so-called moments. Making use of the fact that the corresponding correlation functions are analytic in the complex s -plane except for a cut along the real axis, one can define the moments via [20, 21]

$$R_{V/A}^{w_i}(s_0) = 6\pi i S_{\text{EW}} |V_{ud}|^2 \oint_{|s|=s_0} \frac{ds}{s_0} w_i(s) \left[\Pi_{V/A}^{(1+0)}(s) + \frac{2s}{(s_0 + 2s)} \Pi_{V/A}^{(0)}(s) \right], \quad (4.1)$$

where $\Pi_{V/A}^{(1)}$ and $\Pi_{V/A}^{(0)}$ are spin-1 and spin-0 mesonic correlators, and the particular case of eq. (2.1) corresponds to $R_{V/A}^{w_\tau}(M_\tau^2)$ with the kinematic weight $w_\tau(s) = (1 - s/M_\tau^2)^2(1 + 2s/M_\tau^2)$.

Typically, in the past, moment analyses of τ decay spectra were based on 5 moments to determine 4 parameters, α_s , the gluon condensate $\langle \alpha_s GG \rangle$, as well as the $D = 6$ and $D = 8$ OPE corrections [5, 6, 7, 31]. These analyses, however, suffer from several deficiencies: the used moments are all calculated at $s_0 = M_\tau^2$. In ref. [32] it was then shown that the fit solutions are not stable under a variation of s_0 towards lower values. Next, because pinched-weights with zeros at $s = s_0$ were used, it was assumed that DVs are negligible. Since the DVs enter differently in the different moments, this is potentially dangerous. And finally, the perturbative behaviour of some of the used moments is very bad [16], whence they should be avoided in the α_s analysis. Besides, the ALEPH analyses of refs. [7, 31] were based on data with an incomplete covariance matrix [33], and it remains open what is the impact of this shortcoming.

The aim of refs. [20, 21] was to improve on all the above deficiencies. Firstly, only OPAL data [6] were used, employing the original set in ref. [20] and an updated set incorporating present day τ branching fractions in ref. [21]. Next, DVs were included in the fit which required the use of one un-pinched moment particularly sensitive to DVs, the simplest choice being $w(s) = 1$. In multi-moment fits the additional moments were required to have good perturbative behaviour, which excluded moments sensitive to the gluon condensate, and to suppress $D > 8$ OPE terms. And finally, the s_0 -dependence of the moments was included down to about $s_0 \approx 1.5 \text{ GeV}^2$. Figure 2

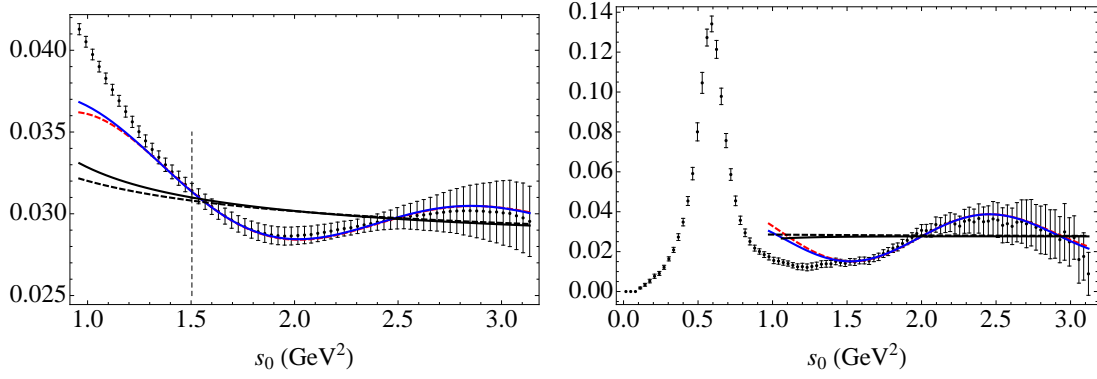


Figure 2: Vector channel fit to the moments $R_V^{w=1}(s_0)$. Left panel: comparison of the experimental moments with the theoretical fit curves for FOPT in blue (solid) and CIPT in red (dashed). The (much flatter) black curves show the pure OPE parts without DVs. Right panel: comparison of the fit prediction for the vector spectral function with the experimental OPAL data.

displays the outcome of the most basic fit with the single moment $w(s) = 1$ and just to the vector-channel spectrum. Also with regard to the OPE, this fit is cleanest as all OPE corrections are additionally suppressed. In ref. [21] the resulting α_s was found to be:

$$\alpha_s(M_\tau) = 0.325(18) \text{ (FOPT)}, \quad \alpha_s(M_\tau) = 0.347(25) \text{ (CIPT)}. \quad (4.2)$$

The uncertainties in (4.2) turn out to be substantially larger than in previous α_s analyses from the τ . One reason is the use of OPAL data, but more importantly employing $w(s) = 1$. Naively one might think that the inclusion of further moments into the fit would reduce the error on α_s . As demonstrated in refs. [20, 21], however, this is not so, because all moments are strongly correlated. Even combined fits to vector and axialvector with up to three weight functions only resulted in a mild reduction in the uncertainty on α_s .

In conclusion, it appears as if only better τ spectral function data, either from a revised ALEPH analysis, or from the B -factories BaBar and Belle, will help to improve the situation, together with a resolution of the CIPT versus FOPT controversy of treating higher perturbative orders.

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